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Existence Results of Steady-States of Semilinear Reaction-Diffusion Equations and Their Applications

C. Y. CHAN*

University of Southwestern Louisiana, Lafayette, Louisiana 70504-1010

AND

MAN KAM KWONG†

Northern Illinois University, DeKalb, Illinois 60115-2888, and Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Illinois 60439-4844

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1. INTRODUCTION

Let $\Omega = (a, b) \times (0, \infty)$. Let us consider the one-dimensional parabolic equation

$$u_t = u_{xx} + f(x, u, u_x) \qquad \text{in } \Omega, \tag{1.1}$$

subject to the initial-boundary conditions on its parabolic boundary $\partial\Omega$,

$$u(x, 0) = g(x) \qquad \text{for} \quad a \leqslant x \leqslant b, \tag{1.2}$$

$$u(a, t) = h(t), \qquad u(b, t) = k(t) \qquad \text{for} \quad t > 0, \tag{1.3}$$

where f: $[a, b] \times (c, d) \times (-\infty, \infty) \to (-\infty, \infty)$ is continuous; a, b, c, and d are constants with $c \geqslant -\infty$, $d \leqslant \infty$; and g, h, and k are continuous functions; $g(a) = h(0)$, and $g(b) = k(0)$. We are interested in the asymptotic

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behavior of the classical solutions. In particular, we would like to know whether a classical solution will approach a steady-state as $t \to \infty$. We remark that in general, however, a solution of (1.1), (1.2), and (1.3) may not exist globally for all $t > 0$. If there is a finite time t_0 at which $\lim_{t \to t_0-} u(x, t) = \infty$, we have a finite-time blow-up. On the other hand, if there is a t_0 at which $\lim_{t \to t_0-} u(x, t) = d < \infty$, causing f to blow up to infinity, we have the phenomenon of quenching.

In Section 2, we quote the well-known Nagumo lemma (cf. Walter [12]) for (1.1). This is used to prove existence of a steady-state solution for the problem (1.1), (1.2), and (1.3) under fairly weak smoothness conditions on f in Section 3. These results are then applied in the rest of the paper to the study of three interesting problems. In Section 4, we extend some results on quenching phenomena by Acker and Walter [1, 2]. In Section 5, existence of the minimal and maximal solutions of boundary value problems of second-order ordinary differential equations is established by embedding them as steady-state solutions of a parabolic equation. This gives an alternative to the monotone iterative methods used, for example, by Ladde, Lakshmikantham, and Vatsala [7]. In Section 6, we give sufficient conditions on the initial data that give rise to the phenomena of finite-time blow-up and the decay of the solution, respectively.

2. Nagumo's Lemma

In the rest of the paper, by a solution of (1.1), we mean a classical C^2 solution.

The classical monograph by Walter [12] has an extensive and excellent account of Nagumo's lemma on parabolic differential inequalities. We quote here a version that suffices for our purpose.

A function $f(x, u, v)$ is said to satisfy a local one-sided Lipschitz condition if, given positive constants c_1 and c_2, there exists a positive constant L such that

$$f(x, u, v) - f(x, w, v) < L(u - w) \tag{2.1}$$

whenever $u > w$, $|u| \leqslant c_1$, $|w| \leqslant c_1$, and $|v| \leqslant c_2$.

Theorem 1. *Suppose that the function f satisfies a local one-sided Lipschitz condition. If $w(x, t)$ is a C^1-function in t, and a C^2-function in x such that*

$$\begin{aligned} w_t \geqslant (\leqslant)\, w_{xx} + f(x, w, w_x) \qquad & \text{in } \Omega, \\ w(x, 0) \geqslant (\leqslant)\, u(x, 0) \qquad & \text{for} \quad a \leqslant x \leqslant b, \\ w(a, t) \geqslant (\leqslant)\, u(a, t),\ w(b, t) \geqslant (\leqslant)\, u(b, t) \qquad & \text{for} \quad t > 0, \end{aligned} \tag{2.2}$$

then $w(x, t) \geqslant (\leqslant) u(x, t)$ in Ω, where $u(x, t)$ is the solution of the initial-boundary value problem (1.1), (1.2), *and* (1.3).

It can be shown that Nagumo's lemma still holds if the local one-sided Lipschitz condition (2.1) is weakened to a condition similar to one used by Kamke in the study of uniqueness of initial value problems of ordinary differential equations (cf. Hartman [5, p. 31]).

As applications of Theorem 1 we give two examples, whose results will be used later on. We now assume that f satisfies the hypotheses of Theorem 1.

EXAMPLE 1. The steady-state equation associated with (1.1) is

$$U''(x) + f(x, U(x), U'(x)) = 0. \tag{2.3}$$

If it has a solution such that for $t > 0$, $U(a) \geqslant h(t)$, and $U(b) \geqslant k(t)$, and if $U(x) \geqslant u(x, 0) = g(x)$, then by Theorem 1, we have $U(x) \geqslant u(x, t)$ in Ω.

EXAMPLE 2. Let us consider the case of zero boundary conditions. If there exists an initial condition $g_1(t) \geqslant 0$ such that its corresponding solution $u_1(x, t)$ tends to zero for all x as t tends to infinity, and if for some t_0, the solution $u \leqslant g_1(x)$, then by Theorem 1,

$$u(x, t) \leqslant u_1(x, t - t_0),$$

and hence $\lim_{t \to \infty} u(x, t) = 0$.

Similarly, in order to show $\lim_{t \to \infty} u(x, t) = \infty$, we need only show that for some $g_1(x)$, the corresponding solution u_1 satisfies $\lim_{t \to \infty} u_1(x, t) = \infty$, and that for some t_0, $u(x, t_0) \geqslant g_1(x)$.

3. STEADY-STATE SOLUTIONS

In this section, we give conditions that guarantee existence of steady-state solutions for (1.1), (1.2), and (1.3). The first lemma is an extension of Theorem 1(a) of Acker and Walter [2].

LEMMA 2. *Let f be such that Nagumo's lemma holds. If $h(t)$ and $k(t)$ are nondecreasing, and*

$$g''(x) + f(x, g(x), g'(x)) \geqslant 0, \tag{3.1}$$

then for each fixed x, $u(x, t)$ is nondecreasing with respect to t.

Proof. Using Theorem 1 on u and g, we obtain $u(x, t) \geq g(x)$ in Ω. Again by using Theorem 1 on $u(x, t+\varepsilon)$ and $u(x, t)$, for any given $\varepsilon > 0$, we have $u(x, t+\varepsilon) \geq u(x, t)$. Since ε is arbitrary, the lemma follows.

Next, we show existence of a steady-state solution.

THEOREM 3. *Suppose that $h(t)$ and $k(t)$ are nondecreasing,* (3.1) *holds,*

$$\sup_{\Omega} u(x, t) < d, \tag{3.2}$$

$$\sup_{\Omega} |u_x(x, t)| < \infty, \tag{3.3}$$

where d is the constant appearing in the domain of definition of f, and f is locally Lipschitz continuous in its second and third variables, then $\lim_{t \to \infty} u(x, t) = U(x)$ exists uniformly, where U is a solution of the steady-state problem (2.3), *with*

$$U(a) = \lim_{t \to \infty} h(t), \qquad U(b) = \lim_{t \to \infty} k(t). \tag{3.4}$$

Proof. By Lemma 2 and (3.2), $\lim_{t \to \infty} u(x, t)$ exists for each x. We need to show that it is a steady-state solution. For each fixed x,

$$\int_0^\infty u_t(x, t)\, dt = \lim_{t \to \infty} u(x, t) - g(x) < \infty.$$

Hence,

$$\int_a^b \int_0^\infty u_t(x, t)\, dt\, dx < \infty.$$

From Lemma 2, $u_t(x, t) \geq 0$. By Fubini's theorem,

$$\int_0^\infty \int_a^b u_t(x, t)\, dx\, dt < \infty. \tag{3.5}$$

Let

$$K(t) = \int_a^b u_t(x, t)\, dx.$$

Then, $K(t)$ is nonnegative, but it need not approach 0 as t tends to ∞. By (3.5), there exists a sequence t_n tending to ∞ such that

$$K(t_n) \text{ tends to } 0. \tag{3.6}$$

Let x_0 be any fixed point in the interval (a, b). From (3.3), the sequence $\{u_x(x_0, t_n)\}$ is bounded, and hence, by passing to a subsequence if necessary, we may assume without loss of generality that its limit, denoted by γ, exists.

Let $V(x)$ be the solution of the initial value problem,

$$V''(x)+f(x, V(x), V'(x))=0, \tag{3.7}$$

$$V(x_0)=\lim_{n\to\infty} u(x_0, t_n)=U(x_0), \qquad V'(x_0)=\gamma. \tag{3.8}$$

The proof of the theorem will be complete if we can show

$$\lim_{n\to\infty} u(x, t_n)=V(x),$$

since then $V(x)\equiv U(x)$. To do so, let us transform (3.7) and (3.8) into a system of first-order integral equations,

$$\bar{V}(x)=\begin{pmatrix} U(x_0) \\ \gamma \end{pmatrix}+\int_{x_0}^{x}\begin{pmatrix} V'(s) \\ -f(s, \bar{V}^T(s)) \end{pmatrix} ds, \tag{3.9}$$

where $\bar{V}^T(x)=(V(x), V'(x))$ is the transpose of $\bar{V}$. Similarly, (1.1) becomes

$$\bar{u}_n(x)=\begin{pmatrix} u(x_0, t_n) \\ u_x(x_0, t_n) \end{pmatrix}+\int_{x_0}^{x}\begin{pmatrix} u_x(s, t_n) \\ -f(s, \bar{u}_n^T(s))+u_t(s, t_n) \end{pmatrix} ds, \tag{3.10}$$

where $\bar{u}_n^T(x)=(u(x, t_n), u_x(x, t_n))$. Since f is locally Lipschitz continuous, there exists a constant c_4 such that

$$|f(s, \bar{V}^T(s))-f(s, \bar{u}_n^T(s)| \leqslant (c_4-1)|\bar{V}(s)-\bar{u}_n(s)|.$$

Let us subtract (3.10) from (3.9), and use the norm, $|\bar{X}|=|X_1|+|X_2|$, where $\bar{X}^T=(X_1, X_2)$. We have

$$|\bar{V}(x)-\bar{u}_n(x)| \leqslant |\bar{V}(x_0)-\bar{u}_n(x_0)|+c_4\int_{x_0}^{x}|\bar{V}(s)-\bar{u}_n(s)|\, ds+K(t_n). \tag{3.11}$$

For any given $\varepsilon>0$, it follows from (3.6) and (3.8) that by choosing n sufficiently large,

$$|\bar{V}(x_0)-\bar{u}_n(x_0)|+K(t_n)<\varepsilon.$$

Using Gronwall's inequality on (3.11), we have

$$|\bar{V}(x)-\bar{u}_n(x)| \leqslant \varepsilon \exp[c_4(b-a)],$$

which implies that as n tends to ∞, $\bar{u}_n(x)$ converges to $\bar{V}(x)$ uniformly with respect to $x\in[a, b]$. This completes the proof of the theorem.

By applying Theorem 3 to $-u(x, t)$, we obtain the following result.

THEOREM 4. *Under hypothesis* (3.3), *if* $h(t)$ *and* $k(t)$ *are nonincreasing, and* f *is locally Lipschitz continuous in its second and third variables,*

$$g''(x)+f(x, g(x), g'(x)) \leqslant 0,$$
$$\inf_{\Omega} u(x, t) > c, \tag{3.12}$$

where c *is the constant appearing in the domain of definition of* f, *then for each fixed* x, $u(x, t)$ *is nonincreasing with respect to* t, *and* $\lim_{t \to \infty} u(x, t) = U(x)$ *exists uniformly.*

We remark that although the proof of Theorem 3 also shows that $\lim_{n \to \infty} u_x(x, t_n) = U'(x)$ uniformly, it is not clear whether $\lim_{t \to \infty} u_x(x, t) = U'(x)$ uniformly. Theorems 3 and 4 are still true if one of the boundary conditions is replaced by Neumann's condition, because such a problem can be extended, by reflection with respect to either $x=a$ or $x=b$, into the form given by (1.1), (1.2), and (1.3).

Below, we give a sufficient condition for (3.3) to hold. This is related to Nagumo's condition in the study of boundary value problems for second-order ordinary differential equations.

THEOREM 5. *If* f *satisfies a local one-sided Lipschitz condition, the boundary values* $h(t)$ *and* $k(t)$ *are both constants,* (3.1) *holds,* u *is bounded, and there exist continuous functions* $r(u)$, *and* $q(u_x^2) > 0$ *such that for all* $|u_x| > \alpha$, *which is a positive constant,*

$$f(x, u, u_x) \leqslant r(u)\, q(u_x^2), \tag{3.13}$$

$$\int_{\alpha^2}^{\infty} \frac{dR}{q(R)} = \infty, \tag{3.14}$$

then (3.3) *holds.*

Proof. By Lemma 2, $u_t \geqslant 0$. Thus,

$$u_{xx}+f(x, u, u_x) \geqslant 0, \tag{3.15}$$

For an arbitrarily fixed t, say t_0, let x_1 be any point in (a, b). If $|u_x(x_1, t_0)| \leqslant \alpha$, then the theorem is proved. Therefore, suppose that either $u_x(x_1, t_0) > \alpha$ or $u_x(x_1, t_0) < -\alpha$. We would like to show that there exists a constant $c_5 > \alpha$ which is independent of x_1 such that $|u_x(x_1, t_0)| < c_5$.

Let us consider the case $u_x(x_1, t_0) > \alpha$ first. At least in a neighborhood to the right of the point x_1, u is strictly increasing. Thus, we have two subcases: $u_x(x, t_0) > 0$ for $x_1 \leqslant x < b$, or there exists a point x_2 (chosen closest to x_1) in (x_1, b) such that $u_x(x_2, t_0) = 0$.

Case 1. $u_x(x, t_0) > 0$ for $x_1 \leqslant x < b$. Let

$$c_6 = \max\{\alpha, -g'(a) + 1, g'(b) + 1\}.$$

There must exist a point x_3 in (x_1, b) such that

$$\frac{g(b) - g(x_3)}{b - x_3} < c_6.$$

By Lemma 2, $u(x, t) \geqslant g(x)$. Since $u(b, t) = g(b)$, it follows that

$$\frac{u(b, t_0) - u(x_3, t_0)}{b - x_3} < c_6.$$

By the Mean Value Theorem, there exists a point x_2 in (x_3, b) such that

$$u_x(x_2, t) = \frac{u(b, t_0) - u(x_3, t_0)}{b - x_3}.$$

If $u_x(x_1, t) \leqslant c_6$, then the theorem is proved; if $u_x(x_1, t) > c_6$, then by applying the Intermediate Value Theorem to u_x, there exists a point x_0 in (x_1, x_2) such that $u_x(x_0, t_0) = c_6$. On the interval $[x_1, x_0]$, u is strictly increasing; we may use u as the independent variable. Let $R = |u_x|^2$. From (3.15),

$$\frac{dR}{du} + 2f(x, u, u_x) \geqslant 0 \qquad \text{for } u(x_1, t_0) \leqslant u \leqslant u(x_0, t_0).$$

By (3.13), $dR/du \geqslant -2r(u)\, q(R)$. Thus,

$$\int_{c_6^2}^{u_x^2(x_1, t_0)} \frac{dR}{q(R)} \leqslant 2 \int_{u(x_1, t_0)}^{u(x_0, t_0)} r(u)\, du. \tag{3.16}$$

Since u is bounded, the right-hand side is bounded. By (3.14), $u_x^2(x_1, t)$ must be bounded, and hence (3.3) holds.

Case 2. $u_x(x_2, t_0) = 0$. As in the previous case, we may apply the Intermediate Value Theorem to u_x to conclude that there exists a point x_0 in (x_1, x_2) such that $u_x(x_0, t_0) = c_6$. The rest of the proof is the same as in Case 1.

Now let us consider the situation $u_x(x_1, t_0) < -\alpha$. In this case, $u(x, t_0)$ is decreasing at least in a neighborhood to the left of x_1. Again, we have two subcases: $u_x(x, t_0) < 0$ for $a < x \leqslant x_1$, or there exists a point x_2 (chosen closest to x_1) in (a, x_1) such that $u_x(x_2, t_0) = 0$. In either case, a similar proof as in $u_x(x_1, t) > \alpha$ shows that there exists a point x_0 in (a, x_1) such that $u_x(x_0, t_0) = -c_6$ and u is strictly decreasing on the interval $[x_0, x_1]$. By using u as the independent variable and R as the dependent variable, it follows, as before, from (3.13), (3.14), and (3.15) that $u_x(x_1, t)$ is bounded.

Similarly, we have the following theorem.

THEOREM 6. *This theorem follows the hypotheses of Theorem 5, except that* (3.1) *and* (3.13) *are replaced respectively by* (3.12) *and*

$$f(x, u, u_x) \geqslant -r(u)\, q(u_x^2). \tag{3.17}$$

Then (3.3) *holds.*

We remark that the hypothesis on the boundedness of u in Theorems 5 and 6 is used to deduce the boundedness of the right-hand side of (3.16). If we impose instead the assumption

$$\int^{\infty} r(u)\, du < \infty,$$

then the boundedness of u follows from the conclusion that u_x is bounded.

As a consequence of Theorem 3, we have the following result, which is used in Sections 4 and 6 in discussing the phenomena of quenching and finite-time blow-up, respectively.

THEOREM 7. *Under the hypothesis of Lemma 2 and* (3.3), *if f is locally Lipschitz continuous in its second and third variables, and the problem* (2.3) *and* (3.4) *has no solution, then either there exists a finite time T such that*

$$\limsup_{t \to T} \sup_x u(x, t) = d < \infty,$$

or

$$\limsup_{t \to \infty} \sup_x u(x, t) = \infty.$$

4. QUENCHING

The terminology quenching was first introduced by Kawarada [6] when he studied the problem

$$u_t = u_{xx} + \frac{1}{1-u} \qquad \text{in } \Omega,$$

with g, h, and k being identically zero. The solution u is said to quench if there exists a finite time T such that

$$\lim_{t \to T^-} \sup\{|u_t(x, t)|: a \leqslant x \leqslant b\} = \infty. \tag{4.1}$$

He proved that there existed a length $b-a$ beyond which

$$\lim_{t \to T^-} \sup_x u(x, t) = 1. \tag{4.2}$$

He claimed that (4.2) implied (4.1). Obviously, (4.1) is a sufficient condition for (4.2). If his claim were true, the two conditions would be equivalent. Thus in studying quenching phenomena, Walter [13] used the necessary condition (4.2). Similarly, Acker and Walter [1, 2] determined the maximum length of the interval beyond which

$$\lim_{t \to T^-} \sup_x u(x, t) = d < \infty, \tag{4.3}$$

where $\lim_{u \to d^-} f = \infty$ when they investigated such phenomena for the more general equations

$$u_t = u_{xx} + f(u), \tag{4.4}$$

$$u_t = u_{xx} + f(u, u_x), \tag{4.5}$$

respectively. Further references on this topic can be found in the survey by Levine [8].

We note that Kawarada's proof that (4.2) implied (4.1) was not entirely correct. Recently, we [3] used a completely different proof to establish the fact that (4.3) implied (4.1) for (4.4) and (4.5), respectively, subject to (1.2) and (1.3). We have thus covered Kawarada's claim as a special case.

Below, we extend the result of Acker and Walter [2] for (4.5) to (1.1) under weaker hypotheses.

THEOREM 8. *Under the hypotheses of Theorem 5, if $f(x, u, u_x)$ is continuously differentiable, f_x is nondecreasing,*

$$f(x, 0, 0) > 0, \tag{4.6}$$

and $\lim_{u \to d^-} f(x, u, u_x) = \infty$ uniformly with respect to x and u_x, then there exists a critical length below which the problem (1.1) with $g(x) \equiv 0$ (and $h(t) \equiv 0 \equiv k(t)$) has a solution for $t > 0$, and beyond which (4.3) holds.

Proof. Let l^* denote the supremum of all values $l = b - a$ such that the steady-state problem (2.3) subject to

$$U(a) = 0 = U(b)$$

has a solution. It follows from Theorem 3 and Example 1 that for $l < l^*$, u exists for $t > 0$. To show that (4.3) occurs for $l > l^*$, let $u(x, t; b)$ denote the solution of (1.1) with zero initial and boundary conditions for all $t > 0$. From (4.6) and (1.1),

$$u_t > u_{xx} + f_3(x, \eta_1, \zeta_1)\, u_x + f_2(x, \eta_1, \zeta_1)\, u,$$

where f_i denotes the partial derivative of f with respect to the ith variable, and η_1 and ζ_1 lie between u and 0, and u_x and 0, respectively. Since f_2 is bounded above, it follows from the strong maximum principle that $u > 0$ in Ω. For any constant $\alpha > 0$, it follows from our hypothesis, $f_x \geqslant 0$, that the function

$$\gamma(x, t) \equiv u(x + \delta, t; b + \alpha) - u(x, t; b) \qquad \text{for} \quad 0 < \delta \leqslant \alpha,$$

satisfies

$$\gamma_t \geqslant \gamma_{xx} + f_3(\xi, \eta, \zeta)\, \gamma_x + f_2(\xi, \eta, \zeta)\, \gamma \qquad \text{in } \Omega,$$

where ξ, η, and ζ lie, respectively, between $x + \delta$ and x, $u(x + \delta, t; b + \alpha)$ and $u(x, t; b)$, and $u_x(x + \delta, t; b + \alpha)$ and $u_x(x, t; b)$. We have made use of the Mean Value Theorem under the assumption that $u(x + \delta, t; b + \alpha)$ exists for $t > 0$. Since $\gamma(x, 0) = 0$, $\gamma(a, t) > 0$, $\gamma(b, t) \geqslant 0$, we have

$$u(x + \delta, t; b + \alpha) > u(x, t; b) \qquad \text{in } \Omega \quad \text{for} \quad 0 < \delta \leqslant \alpha \tag{4.7}$$

by the strong maximum principle. Let us choose positive numbers t_0 and ε such that $0 < \varepsilon < 1$.

$$f(x, z, z_x) \geqslant \alpha^2 + 8\varepsilon/\alpha^2 \qquad \text{for} \quad 0 < d - \varepsilon \leqslant z < d \quad \text{and} \quad -4/\alpha \leqslant z_x \leqslant 4/\alpha,$$

$$u(x(t_0), t_0; b) = d - \varepsilon, \tag{4.8}$$

where $x(t_0)$ is the point at which $u(x, t_0; b)$ attains its maximum. Let us consider the domain $D \equiv (x(t_0), x(t_0) + \alpha) \times (t_0, \infty)$. By (4.7) and Lemma 2, $u(x, t; b + \alpha) \geqslant d - \varepsilon$ on the parabolic boundary ∂D. On the other hand, the function.

$$z(x, t) = d - \varepsilon + [x - x(t_0)]\,[x(t_0) + \alpha - x]\,(t - t_0),$$

attains the value $d - \varepsilon$ on ∂D, and by (4.8),

$$z_{xx} + f(x, z, z_x) \geqslant z_t \qquad \text{in} \quad (x(t_0), x(t_0) + \alpha) \times (t_0, t_0 + 4\varepsilon/\alpha^2) \equiv D_-.$$

By Theorem 1, $z \leqslant u(x, t; b + \alpha)$ on D_-. Since

$$z(x(t_0) + \alpha/2, t_0 + 4\varepsilon/\alpha^2) = d,$$

it follows that $u(x, t; b+\alpha)$ attains d in a finite time. This contradiction proves the theorem.

We note that the above proof does not require the existence of a curve $\phi(t)$ such that $u(x, t; b)$ is monotone increasing in x on $[a, \phi(t)]$ and monotone decreasing in x on $[\phi(t), b]$ (cf. Theorem 1(b) of Acker and Walter). We also note that Acker and Walter required the assumption that there exists a constant $L(B)$ where B is any constant less than d such that whenever $0 \leqslant z \leqslant B$ and $|p| \geqslant 1$, the following inequalities hold,

$$f(z, p) \leqslant L(B)\, p^2, \tag{4.9}$$

$$f_z(z, p) \leqslant L(B)\, |p|, \tag{4.10}$$

$$-pf_p(z, p) \leqslant L(B)\, p^2. \tag{4.11}$$

We do not require assumptions (4.10) and (4.11), and instead of (4.9) we need the weaker hypotheses (3.13) and (3.14).

5. Minimal and Maximal Solutions

Let us consider nonnegative solutions of the problem

$$y'' + f(x, y, y') = 0, \tag{5.1}$$

$$y(a) = 0 = y(b). \tag{5.2}$$

In general, there can be more than one solution, and they need not be ordered in any way. Thus, there may or may not be a minimal or a maximal solution. A common situation in which such a solution exists occurs when f satisfies certain monotonicity conditions, which allows the use of monotone methods to set up successive approximation schemes. Theorems 3 and 4 enable us to establish, respectively, existence of a minimal and a maximal solution without any monotonicity requirement.

Theorem 9. *In addition to the hypothesis of Theorem 5, we assume that* (4.6) *holds and that f is locally Lipschitz continuous in its second and third variables. If the problem* (5.1) *and* (5.2) *has a nonnegative solution, then it has the nonnegative minimal solution.*

Proof. Let us consider the associated problem (1.1) with zero data on its parabolic boundary. By Example 1, $u(x, t) \leqslant y(x)$ for any nonnegative solution y of the problem (5.1) and (5.2). From (4.6), $u \geqslant 0$ in Ω by

Theorem 1. By Theorem 3, $\lim_{t\to\infty} u(x, t)$ exists uniformly and is a solution of the steady-state problem. Hence,

$$\lim_{t\to\infty} u(x, t) \leqslant y(x),$$

and $\lim_{t\to\infty} u(x, t)$ is the minimal solution.

We remark that if $f(x, 0, 0)=0$, then the minimal solution is the trivial solution. Below, we give criteria for existence of the maximal solution.

THEOREM 10. *Under the hypothesis of Theorem 7, if f is locally Lipschitz continuous in its second and third variables, and if all nontrivial solutions y of the problem* (5.1) *and* (5.2) *are bounded by a constant $M>0$, and there exists a solution $Y(x)$ of* (5.1) *with* $\inf_x Y(x) \geqslant M$, *then the problem* (5.1) *and* (5.2) *has the maximal solution.*

Proof. Let $h(t)\equiv Y(a)$, $k(t)\equiv Y(b)$, and $g(x)\equiv Y(x)$. By Theorem 1, the solution u of the problem (1.1), (1.2), and (1.3) is an upper bound of y. By Theorem 5, u_x is bounded. By Theorem 4, u converges to U, and hence the problem (5.1) and (5.2) has the maximal solution.

As an application of the above theorem, we have the following result.

COROLLARY 11. *If f is locally Lipschitz continuous in its second and third variables, and if there exist positive continuous functions $r(y)$ and $q(y'^2)$ such that for all $|y'| \geqslant \alpha$, where α is a positive constant,*

$$f(x, y, y') \leqslant r(y)\, q(y'^2),$$

$$\int_0^\infty r(y)\, dy < \int_0^\infty \frac{dR}{q(R)} = \infty,$$

then the problem (5.1) *and* (5.2) *has the maximal solution.*

Proof. Let y be any solution of (5.1). Using an argument similar to the proof of Theorem 5, we arrive at the inequalities

$$\int_{\alpha^2}^{y'^2(x_1)} \frac{dR}{q(R)} \leqslant 2 \int_{y(x_1)}^{y(x_2)} r(u)\, du < \infty.$$

Thus $|y'(x)| \leqslant c_7$ for some constant c_7.

Now, if y satisfies (5.2), then $|y(x)| \leqslant c_7(b-a)$. On the other hand, if y is a solution of (5.1) subject to the initial conditions, $y(a)=2c_7(b-a)$, $y'(a)=0$, then $y(x) \geqslant c_7(b-a)$. The hypotheses of Theorem 10 are satisfied, and hence the problem (5.1) and (5.2) has the maximal solution.

Our approach also applies to more general boundary conditions and to more general boundedness requirements. A solution $y_m(y_M)$ of the problem (5.1) subject to

$$y(a) = c_8, \qquad y(b) = c_9, \tag{5.3}$$

where c_8 and c_9 are arbitrary constants, is said to be the minimal (maximal) solution relative to a function $y_1(y_2)$ if

$$y_m(x) \geqslant y_1(x), \qquad y_m(x) \leqslant y(x), \qquad (y_M(x) \leqslant y_2(x), y_M(x) \geqslant y(x)) \tag{5.4}$$

for any other solution $y \geqslant y_1 (y \leqslant y_2)$.

The following result can be obtained by using Theorems 1, 5, and 3 for the minimal solution, and Theorems 1, 6, and 4 for the maximal solution since we may choose $h(t)$ and $k(t)$ as nondecreasing (nonincreasing) functions such that $h(0) = y_1(a)$ $(h(0) = y_2(a))$, $k(0) = y_1(b) (k(0) = y_2(b))$, $\lim_{t \to \infty} h(t) = c_8$, and $\lim_{t \to \infty} k(t) = c_9$.

THEOREM 12. *Let f be locally Lipschitz continuous in its second and third variables, and* (5.1) *have two solutions $y_1(x) \leqslant y_2(x)$ such that*

$$y_1(a) \leqslant c_8 \leqslant y_2(a), \qquad y_1(b) \leqslant c_9 \leqslant y_2(b).$$

Under the hypotheses of Lemma 2 and Theorem 5, the minimal solution relative to $y_1(x)$ for the problem (5.1) and (5.3) exists. On the other hand, under the hypotheses of Theorems 6 and 4, the maximal solution relative to y_2 for the problem (5.1) and (5.3) exists. If the minimal and the maximal solutions coincide, then the problem (5.1) and (5.3) has a unique solution y such that $y_1(x) \leqslant y(x) \leqslant y_2(x)$.

6. ASYMPTOTIC DECAY AND BLOW-UP

Let us consider the problem (1.1) subject to

$$u(x, 0) = g(x) \geqslant 0, \qquad u(a, t) = 0 = u(b, t), \tag{6.1}$$

where f is continuous, locally Lipschitz continuous in its second and third variables, and

$$f(x, u, u_x) \geqslant 0, \qquad f(x, 0, 0) = 0. \tag{6.2}$$

A classical prototype is given by

$$f(x, u) = u^s, \qquad s > 1. \tag{6.3}$$

It is well known (cf. Weissler [14], and references quoted there) that if g is large enough, then u may blow up in a finite time, whereas if g is small enough, u is bounded for all t. We would like to show that for a certain class of f, including (6.3), if g is smaller than the unique nontrivial, nonnegative, and bounded solution U of the corresponding steady-state problem

$$U''(x)+f(x, U(x), U'(x))=0, \qquad U(a)=0=U(b), \tag{6.4}$$

then u decays to zero, and if g is larger than U, then blow-up occurs. In particular, our results apply to equations in which the nonlinear term may have a nonconstant coefficient. For the simpler class given by (6.3), similar results are known (see, for instance, Ni *et al.* [11]).

From (6.2), $u \equiv 0$ is a solution of the problem (6.4). We assume that besides this, there is exactly one nontrivial nonnegative solution. This is true in the case of (6.3). It was shown by Coffman [4] that the same is true for some classes of f, including $x^v u^s$, $v>0$, $s>1$, and $a>0$. More recently, Ni [9], and Ni and Nussbaum [10] studied Coffman's problem and extended his results.

LEMMA 13. *Assume that the problem* (6.4) *has a unique nontrivial, nonnegative, and bounded solution U. Let*

$$f(x, \sigma u, \sigma u_x) \leqslant \sigma f(x, u, u_x) \qquad \text{for} \quad 0<\sigma<1. \tag{6.5}$$

(3.17) *and* (3.14) *hold. If for some* $t_0>0$, $u(x, t_0) \leqslant \sigma U(x)$, *then the solution of the problem* (1.1) *and* (6.1) *decays to* 0 *as* t *tends to infinity.*

Proof. By Theorem 1, $u \geqslant 0$ in Ω. From Example 2 of Section 2, it is sufficient to show that the solution $u_1(x, t)$ of the problem (1.1) and (6.1) with $g(x)=\sigma U(x)$ decays to zero as t tends to infinity. From (6.5), g satisfies (3.12). It follows from Theorems 6 and 4 that $\lim_{t \to \infty} u_1(x, t)$ exists. By Theorem 4, $u_1(x, t)$ is nonincreasing with respect to t. Thus, $\lim_{t \to \infty} u_1(x, t)=0$.

Similarly, the following lemma can be proved.

LEMMA 14. *Under the hypotheses of Lemma 13 with* (3.17) *replaced by* (3.13), *if for some* $t_0>0$, $u(x, t_0) \geqslant U(x)/\sigma$, *then the nontrivial solution of the problem* (1.1) *and* (6.1) *becomes unbounded as* t *tends to infinity.*

Below, we give some comparison results.

LEMMA 15. *Assume the problem* (6.4) *has a unique bounded solution U which is positive for* $a<x<b$. *Let*

$$f(x, u, u_x)/u \qquad \text{be nondecreasing in } u. \tag{6.6}$$

If

$$g(x) \leqslant U(x), \qquad g \not\equiv U(x), \tag{6.7}$$

then there exists a constant $c_{10} < 1$ *such that*

$$u(x, t) \leqslant c_{10} U(x) \qquad \textit{for sufficiently large } t. \tag{6.8}$$

On the other hand, if

$$g(x) \geqslant U(x), \qquad g(x) \not\equiv U(x), \tag{6.9}$$

then there exists a constant $c_{11} > 1$ *such that*

$$u(x, t) \geqslant c_{11} U(x) \qquad \textit{for sufficiently large } t.$$

Proof. Let us prove the first part since the proof of the second part is similar. By Theorem 1, $g(x) \leqslant U(x)$ gives $u(x, t) \leqslant U(x)$ for $t > 0$. Let $\Psi(x, t)$ be the solution of the linear problem

$$\Psi_t = \Psi_{xx} + f(x, U, U')\, \Psi/U \qquad \text{in } \Omega. \tag{6.10}$$

$$\Psi(x, 0) = g(x), \qquad \Psi(a, t) = 0 = \Psi(b, t). \tag{6.11}$$

From (6.6), $\Psi_t \geqslant \Psi_{xx} + f(x, u, u_x)\, \Psi/u$. By Theorem 1,

$$u(x, t) \leqslant \Psi(x, t) \qquad \text{in } \Omega. \tag{6.12}$$

Let us solve the problem (6.10) and (6.11) by the method of separation of variables. Let $\{\phi_n\colon n = 1, 2, 3, \ldots\}$ be the set of normalized eigenfunctions of the Sturm–Liouville problem

$$\phi''(x) + f(x, U(x), U'(x))\, \phi(x)/U(x) = -\lambda\phi(x), \qquad \phi(a) = 0 = \phi(b), \tag{6.13}$$

with $\{\lambda_n\colon n = 1, 2, 3, \ldots\}$ as its corresponding eigenvalues. Obviously, zero is an eigenvalue of the problem (6.13) with $U(x)$ as its corresponding eigenfunction. Now, U is positive for $a < x < b$. By the classical Sturm–Liouville theory, $U/\|U\|$, where $\|U\|^2 = \int_a^b U^2(x)\, dx$, is the eigenfunction ϕ_1 corresponding to the smallest eigenvalue $\lambda_1 = 0$. Using the eigenfunction expansion

$$\Psi(x, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n(x),$$

where $g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$, and $a_n = \int_a^b g(x)\, \phi_n(x)\, dx$, we obtain

$$\lim_{t \to \infty} \Psi(x, t) = a_1 \phi_1(x) = \left[\int_a^b g(x)\, U(x)\, dx \right] U(x)/\|U\|^2 < U(x).$$

From (6.12), we have (6.8).

By combining Lemmas 13 to 15, and noting that (6.6) implies (6.5), the following result can be deduced.

THEOREM 16. *Let f be continuous, locally Lipschitz continuous in its second and third variables, and satisfy* (3.14), (3.17), (6.2), *and* (6.6). *Also, assume that the problem* (6.4) *has a unique bounded solution U which is positive for $a<x<b$. If* (6.7) *holds, then* $\lim_{t\to\infty} u(x,t)=0$. *On the other hand, if* (6.9) *holds, then* $\lim_{t\to\infty} u(x,t)=\infty$.

Let us give some more results on unbounded solutions.

THEOREM 17. *Under the hypotheses of Theorem 16, for u to be unbounded, either there is a finite blow-up time, or u exists for $t>0$, and*

$$\lim_{t\to\infty} u(x,t)=\infty \qquad \text{for} \quad a<x<b; \tag{6.14}$$

furthermore, the divergence in (6.14) *is uniform in any proper closed subinterval of* (a, b).

Proof. By Lemma 15, we may assume $u(x,0)=c_{11}U(x)$, *where* $c_{11}>1$. Let us assume that the blow-up time is not finite.

Case 1. $\lim_{t\to\infty}\int_a^b u(x,t)\,U(x)\,dx=\infty$.

Thus, there exists a sequence $\{t_n: n=1,2,3,...\}$ such that

$$b_1 \equiv \frac{1}{\|U\|^2}\int_a^b u(x,t_n)\,U(x)\,dx$$

tends to infinity. By using t_1 as the initial time, a proof similar to that of Lemma 14 gives

$$\lim_{t\to\infty} u(x,t)\geqslant b_1\,U(x).$$

The relation (6.14) and the uniform divergence in any proper closed subinterval then follow.

Case 2. $\lim_{t\to\infty}\int_a^b u(x,t)\,U(x)\,dx<\infty$.

Suppose there is only one point $x_3\in(a,b)$ such that $\lim_{t\to\infty} u(x_3,t)=\infty$. Then, for $x\in(a,x_3)$, $\lim_{t\to\infty} u(x,t)<\infty$. By Theorems 5 and 3, U is a solution of

$$U''(x)+f(x,U(x),U'(x))=0 \qquad \text{for} \quad a<x<x_3.$$

From (6.2), $f(x,U,U')\geqslant 0$, and hence $U''\leqslant 0$, which implies that U is concave downwards for $a\leqslant x\leqslant x_3$. This contradicts $\lim_{x\to x_3} U(x)=\infty$.

Let us suppose that there are two points x_4 and x_5 between a and b with $x_4 < x_5$ such that

$$\lim_{t \to \infty} u(x_4, t) = \infty = \lim_{t \to \infty} u(x_5, t).$$

For any given number $N > 0$, there exists a time t_1 such that $u(x_4, t_1) \geqslant N$, and $u(x_5, t_1) \geqslant N$. By Lemma 2, u is nondecreasing with respect to t. Let us consider the problem $V_t = V_{xx}$ for $x_4 < x < x_5$ and $t_1 < t$, $V(x, t_1) = u(x, t_1)$, $V(x_4, t) = N = V(x_5, t)$ for $t > t_1$. By Theorem 1, $V(x, t) \leqslant u(x, t)$ for $t_1 < t < \infty$. On the other hand, $\lim_{t \to \infty} V(x, t) = N$ for $x_4 < x < x_5$. Thus, $\lim_{t \to \infty} u(x, t) = \infty$ for $x_4 \leqslant x \leqslant x_5$. Hence,

$$\lim_{t \to \infty} \int_{x_4}^{x_5} u(x, t)\, U(x)\, dx = \infty.$$

Thus, Case 2 is really void.

Let us give a result on the finite blow-up time.

THEOREM 18. *Under the hypotheses of Theorem 17, if* $f(x, u, u_x) \geqslant \varepsilon u^s$ *for some positive constants* ε *and* s *with* $s > 1$, *then a finite blow-up time must occur.*

Proof. It follows from Theorem 17 that if a finite blow-up time does not occur, then for any given large constant $N > 0$, it is possible to find a time t_0 such that on any proper closed subinterval $[a_+, b_-]$ of (a, b), $u(x, t) \geqslant N$ for $t > t_0$. By Theorem 1, a solution of the problem,

$$v_t = v_{xx} + \varepsilon v^p \qquad \text{in} \quad [a_+, b_-] \times [t_0, \infty),$$
$$v(x, t_0) = N = v(a_+, t) = v(b_-, t),$$

is a lower bound of u. It is well known (cf. Weissler [14]) that for N sufficiently large, v blows up in a finite time. Hence, the theorem is proved.

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